

Nonlinear n -Pseudo Fermions

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Abstract

Nonlinear pseudo-fermions of degree n (n -pseudo-fermions) are introduced as (pseudo) particles with creation and annihilation operators a and b , $b \neq a^\dagger$, obeying the simple nonlinear anticommutation relation $ab + b^n a^n = 1$. The $(n+1)$ -order nilpotency of these operators follows from the existence of unique (up to a bi-normalization factor) a -vacuum. Supposing appropriate $(n+1)$ -order nilpotent para-Grassmann variables and integration rules the sets of n -pseudo-fermion number states, and 'right' and 'left' ladder operator bi-overcomplete sets of coherent states are constructed. Explicit examples of n -pseudo-fermion ladder operators are provided, and the relation of pseudo-fermions to finite-level pseudo-Hermitian systems is briefly considered.

1 Introduction

In the last decade or so a great deal of attention has been paid in literature to quantum systems with non-Hermitian (quasi-Hermitian, chrypto-Hermitian, PT -symmetric, pseudo-Hermitian) Hamiltonians (see review articles [1, 2] and references therein). More recently a considerable attention is paid to an alternative formalism for description of non-Hermitian systems, based on the concept of the so-called pseudo-bosons (PB) [3]–[10] and pseudo-fermions (PF) [11]–[15]. Pseudo-bosons were originally introduced in [3], where the first bi-overcomplete sets of PB coherent states (CS) is constructed on the example of one-parameter family of PB. Mathematical refinement and further relevant examples of PB are due to Bagarello [4]–[8]. Para-Grassmann CS (nonnormalized ladder-operator CS) for pseudo-Hermitian finite level Hamiltonian systems are constructed and discussed in [14, 15].

The notion of pseudo-Hermitian fermion (phermion) was introduced by Mostafazadeh [11]. Physical example of phermions is given in [12], where (phermions were called pseudo-fermions and) the first bi-overcomplete family of PF coherent states was established. The standard fermions and the pseudo-fermions so far considered [11, 12] are defined through linear in terms of the corresponding creation and annihilation operators anticommutation relations.

In the present paper we introduce *non-linear pseudo-fermions* of degree of nonlinearity n (n -PF), n being positive integer. These are a non-Hermitian extension of the *non-linear fermions* (n -fermions) described in the previous paper [16], and are relevant for description

of finite level non-Hermitian quantum systems. In the next section we provide a brief summary of [16]. In the third section the non-linear pseudo-fermions are introduced and several examples of are presented. In the fourth section 'left' and 'right' ladder operator CS are constructed. The n -PF in finite level pseudo-Hermitian systems are briefly considered in fifth section.

2 Nonlinear n -fermions

The nonlinear n -fermions are defined [16] as particles with annihilation and creation operators $A(n)$ and $A^\dagger(n)$ satisfying the following nonlinear anticommutation relation ¹

$$A(n)A^\dagger(n) + A^{\dagger n}(n)A^n(n) = 1, \quad (1)$$

n being a positive integer. At $n = 1$ the standard fermionic relations $aa^\dagger + a^\dagger a = 1$ are recovered, i.e. $A(1) = a$. In this terminology the standard fermions are "1-fermions", or *linear fermions*.

Supposing the existence of a normalized vacuum state $|0\rangle$ that is annihilated by $A(n)$, one can construct n excited orthonormalized states (Fock states),

$$|k\rangle = A^{\dagger k}(n)|0\rangle, \quad k = 0, 1, \dots, n, \quad (2)$$

and deduce that $A(n)$ are nilpotent of order $n + 1$: $A^{n+1} = 0$. The operators $A(n)$, $A^\dagger(n)$ act on $|k\rangle$ as raising and lowering operators with step 1:

$$A(n)|k\rangle = |k-1\rangle, \quad A^\dagger(n)|k\rangle = |k+1\rangle. \quad (3)$$

The corresponding number operator N , $N|k\rangle = k|k\rangle$, reads

$$N(n) = A^\dagger(n)A(n) + A^{\dagger 2}(n)A^2(n) + \dots + A^{\dagger n}(n)A^n(n), \quad (4)$$

$$[A(n), N(n)] = A(n), \quad [A^\dagger(n), N(n)] = -A^\dagger(n). \quad (5)$$

In this way the state $|k\rangle$, eq. (2), can be regarded as a normalized state with k number of n -fermions, $k = 0, 1, \dots, n$. There are no states with more than n such particles. So the degree of nonlinearity n is the *order of statistics* of our n -fermions. The algebra spanned by $A(n)$, $A^\dagger(n)$ and N , satisfying (1) and (5) could be called *n -fermion algebra*. At $n = 1$ it coincides with the (standard) fermion algebra.

Matrix realization. One can check that the n -fermion algebra (1) admits the following $(n + 1) \times (n + 1)$ matrix representation,

$$A(n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \quad (6)$$

¹Such relation has been suggested (and realized for $n = 2, 3$) in ref. [19] in the context of polynomial relations for the generators of the $su(2)$ Lie algebra.

In order to construct eigenstates of $A(n)$ we adopt the following para-Grassmann algebra for the eigenvalues ζ ,

$$\zeta\zeta^* + \zeta^*\zeta = 0, \quad \zeta^{n+1} = 0 = \zeta^{*n+1}, \quad (7)$$

and the relations

$$\zeta A(n) + A(n)\zeta = 0 = \zeta A^\dagger(n) + A^\dagger(n)\zeta, \quad \zeta|0\rangle = |0\rangle\zeta. \quad (8)$$

These relations are most simple and direct generalizations of those for the standard fermion operators and their Grassmann eigenvalues [20]. They differ from para-Grassmann relations used e.g. in [14, 15, 17]. The para-Grassmann algebra (7) admits $(n+1) \times (n+1)$ matrix representation [16].

Using (7) and (8) the 'right' and 'left' (nonnormalized) eigenstates of $A(n)$ can be constructed [16],

$$\begin{aligned} A(n)|\zeta; n\rangle_r &= \zeta|\zeta; n\rangle_r, \\ |\zeta; n\rangle_r &= \sum_{k=0}^n (-1)^k \zeta^k |k\rangle, \end{aligned} \quad (9)$$

$$\begin{aligned} A(n)|\zeta; n\rangle_l &= |\zeta; n\rangle_l \zeta, \\ |\zeta; n\rangle_l &= \sum_{k=0}^n (-1)^{[\frac{k+1}{2}]} \zeta^k |k\rangle, \end{aligned} \quad (10)$$

The normalized states are $|\zeta; n\rangle_r = N_r |\zeta; n\rangle_r$, $|\zeta; n\rangle_l = N_l |\zeta; n\rangle_l$ where $N_r = N_l = \sqrt{1 - \zeta^* \zeta}$. However, in view of (7), (8), the normalized states $|\zeta; n\rangle_l$, unlike the 'right' $|\zeta; n\rangle_r$, cease being eigenstates of $A(n)$. This unsatisfactory property (and several other ones [16]) of $|\zeta; n\rangle_l$ occurs for other types of para-Grassmann 'left' eigenstates [14, 15, 17, 18] as well.

The sets of both eigenstates $|\zeta; n\rangle_r$ and $|\zeta; n\rangle_l$ can resolve the identity operator,

$$\int d\zeta^* d\zeta |\zeta; n\rangle_r \langle n; \zeta| = 1, \quad (11)$$

$$\int d\zeta^* d\zeta |\zeta; n\rangle_l \langle n; \zeta| = 1, \quad (12)$$

if one adopt the following integration rules [16]

$$\int d\zeta^* d\zeta \zeta^i \zeta^{*k} = \delta_{ik} g_k(n), \quad (13)$$

where $g_k(n)$ are given by

$$g_k(n) = 1 + \sum_{i=1}^{n-k} (-1)^{ki + \frac{i(i+1)}{2}}. \quad (14)$$

For $k = n, n-1, n-2, n-3$ we have

$$g_n = 1, \quad g_{n-1} = 1 + (-1)^n, \quad g_{n-2} = (-1)^{n-1}, \quad g_{n-3} = 0. \quad (15)$$

Note the *different structure* of $g_k(n)$ for odd and even n (i.e. for even and odd dimension of the Hilbert space \mathcal{H}_{n+1}), the structure for odd n being most simple. At $n = 1$ the Berezin rules are reproduced [20]. Thus the states $|\zeta; n\rangle_r$ and $|\zeta; n\rangle_l$ can be qualified as coherent states (CS) – the n -fermion ladder operator CS. At $n = 1$ they reproduce the standard fermionic CS [20]. The n -fermion displacement-operator-like CS can also be constructed, this time the overcompleteness relation needing appropriate weight function $W(\zeta^*\zeta)$ [16].

3 Nonlinear pseudo-fermions

The nonlinear pseudo-fermions of degree n (n -PF) are defined as (pseudo) particles with non-Hermitian annihilation and creation operators $a(n)$, $b(n)$, $b(n) \neq a^\dagger(n)$, satisfying the n -nonlinear anticommutation relation

$$a(n)b(n) + b^n(n)a^n(n) = 1. \quad (16)$$

At $n = 1$ and $b(1) = a^\#(1)$ the phermion relation $aa^\# + a^\#a = 1$ is recovered [11, 12]. In [3] a suggestion is made that if a pseudo-fermion operator $a(1) \equiv a$ admits a vacuum ($a|0\rangle = 0$) then b^\dagger also admits a vacuum and b is η -pseudo adjoint to a , i.e. $b = a^\#$. It appears that this suggestion could be made for $n > 1$ as well. In the above terminology the pseudo-fermions are "1-PFs" (or *linear* PFs).

Suppose that $|\psi_0\rangle$ is annihilated by $a(n)$. Then we construct excited states $|\psi_k\rangle$, $k = 0, 1, \dots$

$$|\psi_k\rangle = b^k(n)|\psi_0\rangle, \quad (17)$$

on which b and a act as raising and lowering operators with step 1,

$$b(n)|\psi_k\rangle = |\psi_{k+1}\rangle, \quad a(n)|\psi_k\rangle = |\psi_{k-1}\rangle. \quad (18)$$

The process is terminated at $k = n$, i.e. $b^{n+1}|\psi_0\rangle = 0$, and this follows from the anticommutation relation (16). Indeed, take $|\psi_{n+1}\rangle := b^{n+1}|\psi_0\rangle$, and multiply it by $a(n)b(n) + b^n(n)a^n(n)$. Using (16)–(18) we obtain

$$|\psi_{n+1}\rangle = (a(n)b(n) + b^n(n)a^n(n))|\psi_{n+1}\rangle = a(n)b(n)|\psi_{n+1}\rangle + b^n(n)a^n(n)|\psi_{n+1}\rangle = 2|\psi_{n+1}\rangle,$$

which is possible iff $|\psi_{n+1}\rangle := b^{n+1}|\psi_0\rangle = 0$. This means that in the space \mathcal{H}_{n+1} spanned by the $n + 1$ vectors $|\psi_k\rangle$ the operators a and b are nilpotent (matrices) of order $n + 1$, i.e. $b^{n+1} = a^{n+1} = 0$. So the set of $|\psi_k\rangle$ is a basis in \mathcal{H}_{n+1} , but since $b \neq a^\dagger$ this basis is not orthogonal.

One can check (using the anticommutation relation (16)) that the states $|\psi_k\rangle$ are eigenstates of the non-Hermitian operator

$$N_{\text{pf}}(n) = b(n)a(n) + b^2(n)a^2(n) + \dots + b^n(n)a^n(n) \quad (19)$$

with eigenvalue k : $N_{\text{pf}}(n)|\psi_k\rangle = k|\psi_k\rangle$. So $N_{\text{pf}}(n)$ plays the role of (non-Hermitian) number operator for n -pseudo-fermions. One can verify that

$$[a(n), N_{\text{pf}}(n)] = a(n), \quad [b(n), N_{\text{pf}}(n)] = -b(n). \quad (20)$$

Since all the $n + 1$ distinct eigenvalues of $N_{\text{pf}}(n)$ are real (nonnegative integers k) the eigenvalues of its Hermitian conjugate $N_{\text{pf}}^\dagger(n)$ are the same (real nonnegative integers k). Denoting the corresponding eigenvectors as $|\varphi_k\rangle$ we write

$$N_{\text{pf}}^\dagger|\varphi_k\rangle = k|\varphi_k\rangle. \quad (21)$$

The nonorthogonal eigenvectors $|\varphi_k\rangle$ can be similarly constructed from the $N_{\text{pf}}^\dagger(n)$ -lower eigenstate $|\varphi_0\rangle$ by means of the raising operator a^\dagger (using $b^\dagger a^\dagger + a^{\dagger n} b^{\dagger n} = 1$ and $[a^\dagger(n), N_{\text{pf}}^\dagger(n)] = -a^\dagger(n)$, $[b^\dagger(n), N_{\text{pf}}^\dagger(n)] = b^\dagger(n)$),

$$|\varphi_k\rangle = a^{\dagger k}(n)|\varphi_0\rangle. \quad (22)$$

It is worth noting at this place that the existence of the nontrivial solution $|\varphi_0\rangle$ of the equation $N_{\text{pf}}^\dagger(n)|\varphi_0\rangle = 0$ follows from the well known property of systems of (here $n + 1$) linear homogeneous algebraic equations $A\vec{x} = 0$: the nontrivial solution \vec{x} exists iff $\det A = 0$. Indeed, if $N_{\text{pf}}(n)|\psi_0\rangle = 0$, $|\psi_0\rangle \neq 0$, then the matrix of $N_{\text{pf}}(n)$ has a vanishing determinant, and so is the case with the determinant of the matrix $N_{\text{pf}}^\dagger(n)$. Therefore the equation $N_{\text{pf}}^\dagger(n)|\varphi_0\rangle = 0$ admits nontrivial solution.

Moreover, the vector $|\varphi_0\rangle$ should be annihilated by the operator b^\dagger : $b^\dagger|\varphi_0\rangle = 0$. This can be easily proven as follows. Applying N_{pf}^\dagger to $b^\dagger|\varphi_0\rangle$ and using $N_{\text{pf}}^\dagger|\varphi_k\rangle = 0$ and $[b^\dagger(n), N_{\text{pf}}^\dagger(n)] = b^\dagger(n)$ we find that $b^\dagger|\varphi_0\rangle$ is an eigenstate of N_{pf}^\dagger with the new eigenvalue -1 . Therefore $b^\dagger|\varphi_0\rangle$ should be orthogonal to all eigenstates $|\psi_k\rangle$ of N_{pf} . The set $\{|\psi_k\rangle\}$ form a basis in \mathcal{H}_{n+1} , therefore $b^\dagger|\varphi_0\rangle = 0$. Thus if $a(n)$ -vacuum $|\psi_0\rangle$ exists, then $b^\dagger(n)$ -vacuum also exists (and vice versa).

The orthogonality of eigenstates $|\psi_i\rangle$ and $|\varphi_j\rangle$ of non-Hermitian operators H and H^\dagger with different real eigenvalues $\varepsilon_i, \varepsilon_j$, used in the above, is a known fact, but for the sake of completeness let us provide here its short proof: $\langle\varphi_j|H^2\psi_i\rangle = \langle\varphi_j|\psi_i\rangle\varepsilon_i^2 = \langle\varphi_j|\psi_i\rangle\varepsilon_i\varepsilon_j$. If $\varepsilon_i \neq \varepsilon_j$ the last equality is possible iff $\langle\varphi_j|\psi_i\rangle = 0$. The states $|\psi_k\rangle$ and $|\varphi_k\rangle$ corresponding to equal eigenvalues can be bi-normalized, so that we have a bi-orthonormalized system of n -pseudo-fermion states,

$$\langle\varphi_k|\psi_j\rangle = \delta_{kj}. \quad (23)$$

Herefrom it follows that $|\psi_j\rangle = \eta|\varphi_j\rangle$, where $\eta = \sum_k |\varphi_k\rangle\langle\varphi_k|$, the inverse operator being $\eta^{-1} = \sum_k |\psi_k\rangle\langle\psi_k|$. Next, one can readily verify (using (23) and the basis $\{|\psi_k\rangle\}$), that the sum $\sum_k |\psi_k\rangle\langle\varphi_k|$ acts on any state $|\psi\rangle$ as unit operator,

$$1 = \sum_k |\psi_k\rangle\langle\varphi_k|. \quad (24)$$

Finally we note that the n -PF creation operator $b(n)$ is η -pseudo-adjoint to $a(n)$: $b(n) = \eta^{-1}a^\dagger(n)\eta =: b^\#$. This can be verified by applying $\eta^{-1}a^\dagger(n)\eta$ to $|\psi_k\rangle$ (the basis vectors in \mathcal{H}_{n+1}) and see that these actions are the same as those of $b(n)$.

Three examples of n -PFs:

(1) $n = 2$.

$$\begin{aligned} a &= \alpha A^\dagger(2) + \beta A^{\dagger^2}(2)A(2), \\ b &= \frac{1}{\alpha + \beta}A(2) + \frac{\beta}{\alpha(\alpha + \beta)}A^2(2)A^\dagger(2), \end{aligned}$$

where $A(2)$, $A^\dagger(2)$ are Hermitian ladder operators of 2-fermions (see section 2). One can check the validity of all the required relations for 2-PF with any β and nonvanishing α , $\alpha \neq -\beta$:

$$\begin{aligned} b &\neq a^\dagger, \\ ab + b^2a^2 &= 1, \quad a^3 = 0 = b^3, \\ a|\psi_0\rangle &= 0 \longrightarrow \langle\psi_0| = (0, 0, p^*), \\ b^\dagger|\varphi_0\rangle &= 0 \longrightarrow \langle\varphi_0| = (0, 0, 1/p), \end{aligned}$$

p being any nonvanishing complex number.

(2) $n = 3$. Any nonvanishing $\alpha, \beta, \gamma, \delta, p$:

$$\begin{aligned} a &= \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta\gamma & 0 & \beta/\delta & -\beta/\delta^2 \\ 0 & 0 & \beta & -\beta/\delta \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1/\gamma\delta & -1/\gamma\delta^2 \\ 1/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\beta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \langle\psi_0| &= (0, 0, p^*, p^*\delta^*), \quad \langle\varphi_0| = (0, 0, 0, 1/p\delta). \end{aligned}$$

(3) $n > 1$. Any nonvanishing $\alpha_i, i = 1, 2, \dots, n, p$:

$$\begin{aligned} a &= \begin{pmatrix} 0 & \alpha_1 & 0 & \dots & 0 \\ 0 & 0 & \alpha_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \dots & \alpha_n \\ 0 & 0 & \cdot & \dots & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_1^{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha_2^{-1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \alpha_n^{-1} & 0 \end{pmatrix}, \\ \langle\psi_0| &= (p^*, 0, \dots, 0), \quad \langle\varphi_0| = (1/p, 0, \dots, 0). \end{aligned}$$

4 n -PF ladder operator eigenstates

The two n -pseudo-fermion lowering and raising ladder operators $a(n)$ and $b(n)$ can be diagonalized using the same para-Grassmann variables ζ as for the n -fermions, eq. (7), and the following relations:

$$\{a(n), \zeta\} = 0 = \{a(n), \zeta^*\}, \quad \{b(n), \zeta\} = 0 = \{b(n), \zeta^*\}, \quad (25)$$

$$\{\zeta, |\psi_0\rangle\} = 0 = \{\zeta^*, |\psi_0\rangle\}, \quad \{\zeta, |\psi_0\rangle\} = 0 = \{\zeta^*, |\psi_0\rangle\}. \quad (26)$$

For the non-bi-normalized 'right' and 'left' eigenstate we find

$$\begin{aligned} a(n)|\zeta; n, \text{pf}\rangle_r &= |\zeta; n, \text{pf}\rangle_r \zeta, \\ |\zeta; n, \text{pf}\rangle_r &= \sum_{k=0} (-1)^k \zeta^k |\psi_k\rangle, \end{aligned} \quad (27)$$

$$\begin{aligned} b^\dagger(n)|\zeta; n, \text{pf}\rangle'_r &= |\zeta; n, \text{pf}\rangle'_r \zeta, \\ |\zeta; n, \text{pf}\rangle'_r &= \sum_{k=0} (-1)^k \zeta^k |\varphi_k\rangle, \end{aligned} \quad (28)$$

$$\begin{aligned} a(n)|\zeta; n, \text{pf}\rangle_l &= \zeta |\zeta; n, \text{pf}\rangle_l, \\ |\zeta; n, \text{pf}\rangle_l &= \sum_{k=0} (-1)^{[\frac{k+1}{2}]} \zeta^k |\psi_k\rangle, \end{aligned} \quad (29)$$

$$\begin{aligned} b^\dagger(n)|\zeta; n, \text{pf}\rangle'_l &= \zeta |\zeta; n, \text{pf}\rangle'_l, \\ |\zeta; n, \text{pf}\rangle'_l &= \sum_{k=0} (-1)^{[\frac{k+1}{2}]} \zeta^k |\varphi_k\rangle. \end{aligned} \quad (30)$$

The bi-normalized states satisfying the relations

$${}_i\langle \text{pf}, n; \zeta | \zeta; n, \text{pf}\rangle'_i = 1, \quad i = l, r, \quad (31)$$

are $|\zeta; n, \text{pf}\rangle_i = \mathcal{N}_i |\zeta; n, \text{pf}\rangle_i$, $|\zeta; n, \text{pf}\rangle'_i = \mathcal{N}'_i |\zeta; n, \text{pf}\rangle'_i$, where $\mathcal{N}_i \mathcal{N}'_i = \sqrt{1 - \zeta^* \zeta}$. If $b = a^\dagger$ they reproduce the normalized n -fermion eigenstates. Note however that the 'left' eigenstates $|\zeta; n, \text{pf}\rangle_l$, $|\zeta; n, \text{pf}\rangle'_l$ when bi-normalized cease being eigenstates of a and b^\dagger . This feature is typical for all 'left' parafermionic eigenstates.

Next we look for bi-overcompleteness relations. It turned out that the families of both 'left' and 'right' n -pseudo-fermion eigenstates form the bi-overcomplete sets of states with respect to the same new integration relations (13), (14). Instead of (11), (12) we now have the bi-overcompleteness relations

$$\int d\zeta^* d\zeta |\zeta; n, \text{pf}\rangle'_r {}_r\langle \text{pf}, n; \zeta | = 1, \quad (32)$$

$$\int d\zeta^* d\zeta |\zeta; n, \text{pf}\rangle'_l {}_l\langle \text{pf}, n; \zeta | = 1. \quad (33)$$

In view of the above bi-overcompleteness relations the two sets $\{|\zeta; n, \text{pf}\rangle_r, |\zeta; n, \text{pf}\rangle'_r\}$ and $\{|\zeta; n, \text{pf}\rangle_l, |\zeta; n, \text{pf}\rangle'_l\}$ can be qualified as n -pseudo-fermion 'left' and 'right' ladder operator coherent states (CS). At $n = 1$ they recover the bi-overcomplete sets of pseudo-fermionic CS [12] (1-pseudo-fermion, or linear pseudo-fermion CS in the present terminology).

5 n -PF and non-Hermitian systems

In this section we briefly consider the relations of n -pseudo-fermions with finite level (non-Hermitian) quantum systems. For Hermitian systems similar relations are discussed in [16].

Consider a system with finite number of non-degenerate (possibly not equidistant) 'energy' levels $(n+1)$ levels) ε_k , $k = 0, 1, \dots, n$, and let $|\psi_k\rangle$ be the corresponding wave functions. Denote the finite dimensional Hilbert space spanned by $|\psi_k\rangle$ as \mathcal{H}_{n+1} . If $|\psi_k\rangle$ are not orthogonal to each other they could be regarded as eigenstates of some non-Hermitian Hamiltonian H : $H|\psi_k\rangle = \varepsilon_k|\psi_k\rangle$. In a more general setting the eigenvalues ε_k may be complex quantities. If ε_k are real or come in complex conjugate pairs then H is pseudo-Hermitian [2]: $H = \eta^{-1}H^\dagger\eta =: H^\#$, where η is some Hermitian operator. For H^\dagger the eigenvalue relations are $H^\dagger|\varphi_k\rangle = \varepsilon_k^*|\varphi_k\rangle$. For different k, k' the states $|\psi_k\rangle$ and $|\varphi_{k'}\rangle$ are orthogonal and for equal k they can be bi-normalized, $\langle\psi_k|\varphi_{k'}\rangle = \delta_{kk'}$.

The general lowering and raising operators between levels are of the form [14]

$$a(n; \vec{\rho}) = \sum_{k=0}^{n-1} \sqrt{\rho_k} |\psi_k\rangle \langle \varphi_{k+1}|, \quad b(n; \vec{\rho}) = \sum_{k=0}^{n-1} \sqrt{\rho_k} |\psi_{k+1}\rangle \langle \varphi_k|. \quad (34)$$

where ρ_k are arbitrary complex (dimensionless) quantities. Such operators with $\rho_k = [[k+1]] = (q^{k+1} - q^{-k-1})/(q - 1/q) \equiv \rho_k^{(q)}$, $q = \exp(i\pi/(n+1))$ were considered in [15]. At $\rho_k = \varepsilon_{k+1}$ we find that $b(n; \vec{\varepsilon})a(n; \vec{\varepsilon})|\psi_k\rangle = \varepsilon_k|\psi_k\rangle$, which means that these operators factorize the Hamiltonian,

$$H = b(n; \vec{\varepsilon})a(n; \vec{\varepsilon}). \quad (35)$$

If $\varepsilon_0 \neq 0$ then $H = b(n; \vec{\varepsilon})a(n; \vec{\varepsilon}) + \varepsilon_0$. Note that here H is dimensionless, along with ε_k and operators $a(n)$ and $b(n)$.

At $\rho_k = 1$ the operators $a(n; 1)$, $b(n; 1)$ obey the relation (16), i.e. $a(n; 1)$, $b(n; 1)$ are n -pseudo-fermion ladder operators for the $(n+1)$ -level quantum system: $a(n; 1) = a(n)$, $b(n; 1) = b(n)$. The general ladder operators $a(n; \vec{\rho})$, $b(n; \vec{\rho})$ can be expressed in terms of $a(n)$, $b(n)$ as

$$a(n; \vec{\rho}) = \sigma_0 a(n) + (\sigma_1 - \sigma_0) b(n) a^2(n) + \dots + (\sigma_{n-1} - \sigma_{n-2}) b^{n-1}(n) a^n(n), \quad (36)$$

$$b(n; \vec{\rho}) = \sigma_0 b(n) + (\sigma_1 - \sigma_0) b^2(n) a(n) + \dots + (\sigma_{n-1} - \sigma_{n-2}) b^n(n) a^{n-1}(n). \quad (37)$$

where, to shorten the equations we have put $\sqrt{\rho_k} \equiv \sigma_k$.

It is not difficult to check that H commutes with the n -PF number operator $N_{\text{pf}}(n) = b(n)a(n) + \dots + b^n(n)a^n(n)$. In view also of the fact that $|\psi_k\rangle$ are eigenstates of $N_{\text{pf}}(n)$ with eigenvalues k one could interpret the energy value ε_k as a sum of the energies ε_k/k of k number of n -pseudo-fermions. If the spectrum of H is equidistant, then H is proportional to $N_{\text{pf}}(n)$.

Following the scheme developed in section 3 (and in [16] for Hermitian case) one can construct bi-overcomplete 'left' and 'right' ladder operator eigenstates for $a(n; \vec{\rho})$ and $b^\dagger(n; \vec{\rho})$ using the para-Grassmann algebra (7) and the integration rules (13). Bi-overcomplete sets of para-Grassmann 'left' (nonnormalized) eigenstates of $a(n; \vec{\rho})$ and $b^\dagger(n; \vec{\rho})$ were constructed in [14] using different paragrassmannian variables and integration rules. In [15] overcomplete families of eigenstates of $a' = q^{N/2}a(n; \vec{\rho}^{(q)})$ (where, in our notations, $[[N]] =$

$b(n; \vec{\rho}^{(q)})a(n; \vec{\rho}^{(q)})$ and $c' = q^{N'/2}b^\dagger(n; \vec{\rho}^{(q)})$ (where $[[N']] = a^\dagger(n; \vec{\rho}^{(q)})b^\dagger(n; \vec{\rho}^{(q)})$ are built up using also different paragrassmannian variables and integration rules.

Conclusion

We have introduced nonlinear pseudo-fermions of degree n (n -pseudo-fermions) as (pseudo) particles with non-Hermitian annihilation and creation operators $a(n)$, $b(n)$, $b(n) \neq a^\dagger(n)$, satisfying the n -nonlinear anticommutation relation $a(n)b(n) + b^n(n)a^n(n) = 1$. A pair of non-Hermitian operators a , $b \neq a^\dagger$ could represent n -pseudo-fermion if they obey the relation $ab + b^n a^n = 1$ and a admits a nontrivial ground state $|\psi_0\rangle$, $a|\psi_0\rangle = 0$. Then b^\dagger -vacuum also exists, a and b are nilpotent of order $n + 1$, and bi-orthonormalized set of Fock states and bi-overcomplete sets of 'left' and 'right' CS can be constructed, as we have done this in the paper, using appropriately defined new paragrassmannian variables and integration rules. At $b = a^\dagger$ (the Hermitian case) n -pseudo-fermion CS recover the n -fermion CS [16], and at $n = 1$ both 'left' and 'right' n -pseudo-fermion CS reproduce the pseudo-fermionic CS of ref. [12]. Different kind of (nonnormalized) para-Grassmann 'left' CS for finite level pseudo-Hermitian systems are considered in [14, 15].

Three different families of n -pseudo-fermion operators have been provided as examples for $n = 2$, $n = 3$ and any $n > 1$. The n -pseudo-fermion operators can be introduced for any pseudo-Hermitian system with non-degenerate finite number of energy levels ε_k . The n -pseudo-fermion number operator commutes with the corresponding non-Hermitian Hamiltonian and thereby the energy ε_k could be regarded as a sum of energies of k number of pseudo-particle (or pseudo-excitations). If the energy levels are equidistant then the Hamiltonian is proportional to the pseudo-fermion number operator.

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